CEX CLASA A VII-A

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profesor Timohe Gabriel, Colegiul National “G. Ibraileanu” Iasi

Multimea numerelor reale.

Probleme de ordonare

Exercitii propuse:

I.

1.Aratati ca daca $\in N , n\geq 2$ , atunci $\frac{1}{2}<\frac{1}{n+1}+\frac{1}{n+2}+…+\frac{1}{2n}<\frac{3}{4}$ .

Solutie:

$$\frac{1}{n+1} , …, \frac{1}{2n-1}> \frac{1}{2n} ⇒\frac{1}{2}<\frac{1}{n+1}+\frac{1}{n+2}+…+\frac{1}{2n}$$

$$\frac{1}{n+1}+\frac{1}{n+2}+…+\frac{1}{2n}=\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+…+\frac{1}{2n}-\frac{1}{n}=$$

$$=\frac{1}{2}\left[\left(\frac{1}{n}+\frac{1}{2n}\right)+\left(\frac{1}{n+1}+\frac{1}{2n-1}\right)+…+\left(\frac{1}{2n}+\frac{1}{n}\right)\right]-\frac{1}{n}=$$

$$=\frac{1}{2}\left[\frac{3n}{2n^{2}}+\frac{3n}{\left(n+1\right)\left(2n-1\right)}+…+\frac{3n}{2n^{2}}\right]-\frac{1}{n}\leq \frac{1}{2}\left(\frac{3n}{2n^{2}}+…+\frac{3n}{2n^{2}}\right)-\frac{1}{n}=\frac{3}{4}-\frac{1}{4n}<\frac{3}{4}$$

2.Aratati ca pentru orice $n\in N , n\geq 2$ exista $x\_{1},x\_{2},…,x\_{n}\in N^{\*}$ astfel incat $x\_{1}\leq x\_{2}\leq …\leq x\_{n}$ si $x\_{1}+x\_{2}+…$+$x\_{n}=x\_{1}x\_{2}∙…∙x\_{n}$ . Demonstrati ca pentru $n\in \left\{2,3,4\right\}$ numerele $x\_{1},x\_{2},…,x\_{n}$ sunt unic determinate.

Solutie:

$x\_{1}=x\_{2}=…=x\_{n-2}=1 , x\_{n-1}=2 , x\_{n}=n$ solutie.

$$n=2⇒\left(x\_{1}-1\right)\left(x\_{2}-1\right)=1⇒\left(x\_{1}-1\right)=1 , \left(x\_{2}-1\right)=1$$

$$n=3⇒x\_{1}+x\_{2}+x\_{3}=x\_{1}x\_{2}x\_{3}$$

Daca $x\_{1}\geq 2⇒x\_{1}x\_{2}x\_{3}\geq 2x\_{2}x\_{3}\geq 4x\_{3}>3x\_{3}\geq x\_{1}+x\_{2}+x\_{3}⇒x\_{1}=1$. Rezulta ca $\left(x\_{2}-1\right)\left(x\_{3}-1\right)=2⇒\left(x\_{2}-1\right)=1 , \left(x\_{3}-1\right)=2$

$$n=4⇒x\_{1}+x\_{2}+x\_{3}+x\_{4}=x\_{1}x\_{2}x\_{3}x\_{4}⇒x\_{1}=1 , x\_{2}=1 , \left(x\_{3}-1\right)=1 , \left(x\_{4}-1\right)=3$$

3.Aratati ca $=\frac{1}{2}∙\frac{3}{4}∙\frac{5}{6}∙…∙\frac{99}{100}<\frac{1}{10}$ .

Solutie:

$$b=\frac{2}{3}∙\frac{4}{3}∙\frac{6}{5}∙…∙\frac{98}{99} ⇒ab=\frac{1}{100}$$

$$\left(2n-1\right)\left(2n+1\right)<2n2n⇒\frac{2n-1}{2n}<\frac{2n}{2n+1}⇒\frac{1}{2}<\frac{2}{3} , \frac{3}{4}<\frac{4}{5} ,…, \frac{99}{100}<1⇒0<a<b⇒$$

$$a^{2}<ab⇒a^{2}<\frac{1}{100}$$

4.Daca $x\_{1},x\_{2},…,x\_{n}$ sunt numere naturale distincte mai mari sau egale cu 2, atunci aratati ca

$\left(1-\frac{1}{x\_{1}^{2}}\right)\left(1-\frac{1}{x\_{2}^{2}}\right)∙…∙\left(1-\frac{1}{x\_{n}^{2}}\right)>\frac{1}{2}$ .

Solutie:

$$x\_{1}<x\_{2}<…<x\_{n}⇒x\_{k}\geq k+1⇒1-\frac{1}{x\_{k}^{2}}\geq \frac{k(k+2)}{(k+1)^{2}}$$

$$\left(1-\frac{1}{x\_{1}^{2}}\right)\left(1-\frac{1}{x\_{2}^{2}}\right)∙…∙\left(1-\frac{1}{x\_{n}^{2}}\right)\geq \frac{1∙3}{2^{2}}∙\frac{2∙4}{3^{2}}∙…∙\frac{n(n+2)}{(n+1)^{2}}=\frac{1}{2}\left(1+\frac{1}{n+1}\right)>\frac{1}{2}$$

5.Daca $x\_{1},x\_{2},…,x\_{n}$ sunt numere naturale nenule distincte ($n\geq 2$), atunci aratati ca

$\frac{1}{x\_{1}^{2}}+\frac{1}{x\_{2}^{2}}+…+\frac{1}{x\_{n}^{2}} \ne 1$ .

Solutie:

$$x\_{1}<x\_{2}<…<x\_{n}$$

$$x\_{1}=1⇒\frac{1}{x\_{1}^{2}}+\frac{1}{x\_{2}^{2}}+…+\frac{1}{x\_{n}^{2}}>1$$

$$x\_{1}\geq 2⇒x\_{k}\geq k+1⇒\frac{1}{x\_{1}^{2}}+\frac{1}{x\_{2}^{2}}+…+\frac{1}{x\_{n}^{2}}\leq \frac{1}{2^{2}}+\frac{1}{3^{2}}+…∙\frac{1}{\left(n+1\right)^{2}}<$$

$$<\frac{1}{1∙2}+…+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

6.Fie $x\in R$. Aranjati in ordine crescatoare numerele

$a\_{1}=\frac{1}{x^{2}+x+1} , a\_{2}=\frac{2}{x^{2}+x+2} , …, a\_{10}=\frac{10}{x^{2}+x+10}$ .

Solutie:

$$x\in \left\{-1,0\right\}⇒x^{2}+x=0⇒a\_{1}=…=a\_{10}$$

$$x\in R\\left\{-1,0\right\}⇒x^{2}+x=y$$

Daca $y>0⇔x<-1 sau x>0$ atunci

$$y>\frac{y}{2}>\frac{y}{3}>…>\frac{y}{10}>0⇒ y+1>\frac{y+2}{2}>\frac{y+3}{3}>…>\frac{y+10}{10}>0⇒$$

$$⇒a\_{1}<…<a\_{10}$$

Daca $<0⇔x>-1 si x<0$ , atunci din $\left(x+\frac{1}{2}\right)^{2}\geq 0$ rezulta $-\frac{1}{4}\leq y<0⇒$

$$⇒-\frac{1}{4}\leq y<\frac{y}{2}<…<\frac{y}{10}⇒\frac{3}{4}\leq y+1<\frac{y+2}{2}<…<\frac{y+10}{10}⇒a\_{10}<…<a\_{1}$$

7.Fie $x,y,z$ sunt numere reale nenule. Aratati ca daca $xyz=1 si x+y+z>\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ atunci unul dintre numerele $x,y,z$ este mai mare decat 1, iar celelalte doua sunt mai mici decat 1.

Solutie:

$$xyz=1⇒nu toate numerele sunt>1$$

$$z=\frac{1}{xy}⇒x+y+\frac{1}{xy}>\frac{1}{x}+\frac{1}{y}+xy \left(+1\right)⇒\left(x-1\right)\left(y-1\right)<\left(\frac{1}{x}-1\right)\left(\frac{1}{y}-1\right)=$$

$$=\frac{\left(x-1\right)(y-1)}{xy}=\left(x-1\right)\left(y-1\right)z⇒\left(x-1\right)\left(y-1\right)\left(z-1\right)>0$$

8.Se dau numerele $x\_{1},x\_{2},…,x\_{100}$ reale pozitive astfel incat $x\_{1}+x\_{2}+…+x\_{100}<3$ si $x\_{1}^{2}+x\_{2}^{2}+…+x\_{100}^{2}>1$. Aratati ca printer ele se gasesc totdeauna trei numere a caror suma este mai mare decat 1.

Solutie:

Daca unul din numere este mai mare sau egal cu 1 atunci problema este rezolvata.

Daca toate numerele sunt mai mici decat 1 si $x\_{100}\leq …\leq x\_{2}\leq x\_{1}$ atunci

$$x\_{1}+x\_{2}+x\_{3}\geq x\_{1}+x\_{2}+x\_{3}-\left(x\_{1}-x\_{3}\right)\left(1-x\_{1}\right)-\left(x\_{2}-x\_{3}\right)\left(1-x\_{2}\right)=$$

$$=x\_{1}^{2}+x\_{2}^{2}+x\_{3}\left(3-x\_{1}-x\_{2}\right)>x\_{1}^{2}+x\_{2}^{2}+x\_{3}\left(x\_{3}+x\_{4}+…+x\_{100}\right)\geq $$

$$\geq x\_{1}^{2}+x\_{2}^{2}+…+x\_{100}^{2}>1$$

9.Aratati ca pentru orice numar natural nenul $n$ are loc inegalitatea

$1+\frac{1}{2}+\frac{1}{3}+…+\frac{1}{n}\geq \frac{2n}{n+1}$ .

Solutie:

Inegalitatea se verifica pentru $n\in \left\{1,2,3\right\}$.

Pentru $n\geq 4⇒1+\frac{1}{2}+\frac{1}{3}+…+\frac{1}{n}\geq 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=\frac{25}{12}>2 si \frac{2n}{n+1}<2$.

II.

1.Sa se determine numerele reale $x$pentru care are loc egalitatea:

$\sqrt{\frac{x-7}{1989}}+\sqrt{\frac{x-6}{1990}}+\sqrt{\frac{x-5}{1991}}=\sqrt{\frac{x-1989}{7}}+\sqrt{\frac{x-1990}{6}}+\sqrt{\frac{x-1991}{5}}$ .

Solutie:

$$x\geq 1991$$

$$\sqrt{1-\frac{-x+1996}{1989}}+\sqrt{1-\frac{-x+1996}{1990}}+\sqrt{1-\frac{-x+1996}{1991}}=$$

$$=\sqrt{1-\frac{-x+1996}{7}}+\sqrt{1-\frac{-x+1996}{6}}+\sqrt{1-\frac{-x+1996}{5}}$$

$$x<1996⇒1-\frac{-x+1996}{1989}>1-\frac{-x+1996}{7},…⇒S>D⇒nu solutii$$

$$x>1996⇒1-\frac{-x+1996}{1989}<1-\frac{-x+1996}{7},…⇒S<D⇒nu solutii$$

Solutia este $x=1996$.

2.Fie $n\in N , n\geq 3$. Demonstrati ca

 $n-2<\sqrt{1+\frac{1}{2^{2}}+\frac{1}{3^{2}}}+\sqrt{1+\frac{1}{3^{2}}+\frac{1}{4^{2}}}+…+\sqrt{1+\frac{1}{\left(n-1\right)^{2}}+\frac{1}{n^{2}}}<n-\frac{1}{n}$ .

Solutie:

$$1+\frac{1}{\left(k-1\right)^{2}}+\frac{1}{k^{2}}=\left(\frac{k^{2}-k+1}{(k-1)k}\right)^{2}=\left(1+\frac{1}{(k-1)k}\right)^{2}=1+\frac{1}{k-1}-\frac{1}{k}$$

$$\sqrt{1+\frac{1}{2^{2}}+\frac{1}{3^{2}}}+\sqrt{1+\frac{1}{3^{2}}+\frac{1}{4^{2}}}+…+\sqrt{1+\frac{1}{\left(n-1\right)^{2}}+\frac{1}{n^{2}}}=1+1+…+1+\frac{1}{2}-\frac{1}{n}=n-\frac{1}{n}-\frac{3}{2}$$

3.Se considera numarul natural nenul $n$ si numerele

$$x=\left(\sqrt{2}-\sqrt{1}\right)\left(\sqrt{3}+\sqrt{2}\right)\left(\sqrt{4}-\sqrt{3}\right)∙…∙\left(\sqrt{2n+1}+\sqrt{2n}\right) si$$

$y=\left(\sqrt{2}+\sqrt{1}\right)\left(\sqrt{3}-\sqrt{2}\right)\left(\sqrt{4}+\sqrt{3}\right)∙…∙\left(\sqrt{2n+1}-\sqrt{2n}\right)$.

Sa se arate ca $x>y , x+y>2 si \frac{x}{y}<n+1$.

Solutie:

$$xy=1$$

$$x=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{1}}∙\frac{\sqrt{5}+\sqrt{4}}{\sqrt{4}+\sqrt{3}}∙…∙\frac{\sqrt{2n+1}+\sqrt{2n}}{\sqrt{2n}+\sqrt{2n-1}}>1⇒y<1$$

$$\frac{x+y}{2}>\sqrt{xy}$$

$$\frac{x}{y}=\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}+\sqrt{1}}\right)^{2}∙\left(\frac{\sqrt{5}+\sqrt{4}}{\sqrt{4}+\sqrt{3}}\right)^{2}∙…∙\left(\frac{\sqrt{2n+1}+\sqrt{2n}}{\sqrt{2n}+\sqrt{2n-1}}\right)^{2}$$

$$\frac{\sqrt{2n+1}+\sqrt{2n}}{\sqrt{2n}+\sqrt{2n-1}}<\frac{\sqrt{n+1}}{\sqrt{n}}$$

$$\frac{x}{y}<\frac{2}{1}∙\frac{3}{2}∙…∙\frac{n+1}{n}=n+1$$

4.Se da $n$ un numar natural impar si numerele

$$A=\frac{1}{1+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+…+\frac{1}{\sqrt{n^{2}-2}+\sqrt{n^{2}-1}} si$$

$B=\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{4}+\sqrt{5}}+…+\frac{1}{\sqrt{n^{2}-1}+\sqrt{n^{2}}}$ .

Demonstrati ca $0<A-B<1$ si determinate partea intreaga a numarului .

Solutie:

$$\frac{1}{1+\sqrt{2}}>\frac{1}{\sqrt{2}+\sqrt{3}} , … ⇒A>B$$

$A-B=\frac{1}{1+\sqrt{2}}-\left(\frac{1}{\sqrt{2}+\sqrt{3}}-\frac{1}{\sqrt{3}+\sqrt{4}}\right)-…-\frac{1}{\sqrt{n^{2}-1}+\sqrt{n^{2}}}$ <$\frac{1}{1+\sqrt{2}}<1$

$$A+B=\sqrt{2}-1+\sqrt{4}-\sqrt{3}+…=\sqrt{n^{2}}-1=n-1$$

$$\frac{n-1}{2}<A=\frac{A+B}{2}+\frac{A-B}{2}=\frac{n-1}{2}+\frac{A-B}{2}<\frac{n-1}{2}+\frac{1}{2}⇒\left[A\right]=\frac{n-1}{2}$$

III. Exercitii propuse ca tema pentru elevi

1.Determinati numarul natural $n$ din egalitatea:

$\frac{1}{1∙3}+\frac{2}{3∙7}+\frac{2^{2}}{7∙15}+\frac{2^{3}}{15∙31}+…+\frac{2^{n-1}}{\left(2^{n}-1\right)\left(2^{n+1}-1\right)}=\frac{2^{2003}-1}{2^{2004}}$ .

Indicatie:

$$\frac{2^{n}}{\left(2^{n}-1\right)\left(2^{n+1}-1\right)}=\frac{1}{2^{n}-1}-\frac{1}{2^{n+1}-1}$$

Se obtine $n=2003$.

2.Se dau numerele $=\frac{1}{10∙11^{2}}+\frac{1}{11∙12^{2}}+\frac{1}{12∙13^{2}}+…+\frac{1}{19∙20^{2}}$ , $B=\frac{1}{11∙10^{2}}+\frac{1}{12∙11^{2}}+\frac{1}{13∙12^{2}}+…+\frac{1}{20∙19^{2}}$ .

Demonstrati ca:

a)$A+B=\frac{3}{400}$

b)$\frac{1}{2}\left(\frac{1}{10∙11}-\frac{1}{20∙21}\right)<A<\frac{3}{800}<B<\frac{1}{2}\left(\frac{1}{9∙10}-\frac{1}{19∙20}\right)$.

3.Fie $x,y,z,t$ numere rationale pozitive. Aratati ca daca $\frac{x}{y+z+t}=\frac{y}{z+t+x}=\frac{z}{t+x+y}=\frac{t}{x+y+z}$ atunci

$\frac{x+y}{z+t}+\frac{y+z}{t+x}+\frac{z+t}{x+y}+\frac{t+x}{y+z}=4$ .

Indicatie:

$$\frac{x}{y+z+t}=\frac{y}{z+t+x}=\frac{z}{t+x+y}=\frac{t}{x+y+z}=\frac{x+y+z+t}{3(x+y+z+t)}⇒$$

$\frac{x}{y+z+t}=\frac{y}{z+t+x}=\frac{1}{3}⇒\frac{x+y}{x+y+2(z+t)}=\frac{1}{3}⇒\frac{x+y}{2(z+t)}=\frac{1}{2}$ .

4.Daca $x,y,z$ sunt numere reale pozitive cu $x+2y+4z=1$ atunci aratati ca $\sqrt{2x+1}+\sqrt{4y+3}+\sqrt{8z+5}<7$.

Indicatie:

$2x+1=3-\left(4y+8z\right)⇒\sqrt{2x+1}<\sqrt{3}<1,8$.

5.Se considera numarul $a=\frac{1}{2^{2}}+\frac{1}{3^{2}}+…+\frac{1}{100^{2}}$ . Demonstrati ca $0,2<\sqrt{\frac{a}{11}}<0,3$.

Indicatie:

$$n\left(n-1\right)<n^{2}<n\left(n+1\right)⇒\frac{1}{n}-\frac{1}{n+1}<\frac{1}{n^{2}}<\frac{1}{n-1}-\frac{1}{n}⇒$$

$$⇒\frac{1}{2}-\frac{1}{101}<a<1-\frac{1}{100}$$