CEX CLASA A VII-A

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Multimea numerelor reale.

Calcul algebric

Exercitii propuse:

1. Daca $a,b,c,d\in N^{\*} si x,y\in Q$ sa se arate ca:

a)$x\sqrt{2}+y\sqrt{3}\in Q daca si numai daca x=y=0$;

b)Daca $\frac{a\sqrt{3}+b\sqrt{2}+c}{b\sqrt{3}+c\sqrt{2}+d}\in Q$ atunci $a+d\geq b+c si \left(a^{2}+b^{2}-c^{2}-d^{2}\right)\vdots \left(a+b+c+d\right)$.

Solutie:

a)$x\sqrt{2}+y\sqrt{3}=r\in Q⇒2\sqrt{6}xy=r^{2}-2x^{2}-3y^{2}\in Q$

$$xy\ne 0⇒\sqrt{6}\in Q (F)$$

$$xy=0⇒x=0 sau y=0⇒y\sqrt{3}\in Q sau x\sqrt{2}\in Q (F)$$

b)$ \frac{a\sqrt{3}+b\sqrt{2}+c}{b\sqrt{3}+c\sqrt{2}+d}=r>0⇒\left(a-rb\right)\sqrt{3}+\left(b-rc\right)\sqrt{2}=dr-c\in Q⇒a=rb,b=rc,c=rd⇒$

$$⇒b=r^{2}d,a=r^{3}d⇒a+d-b-c=d\left(r^{3}+1\right)-rd\left(r+1\right)=(r+1)d\left(r-1\right)^{2}\geq 0$$

$a^{2}+b^{2}-c^{2}-d^{2}=d^{2}\left(r^{2}+1\right)^{2}\left(r^{2}-1\right)\vdots \left(a+b+c+d\right)=d(r+1)\left(r^{2}+1\right)$.

2. Daca $a,b,c\in R$ astfel incat $ab+bc+ca=3$ , aratati ca:

$\sqrt{a^{4}+2b^{2}c^{2}+1}+\sqrt{b^{4}+2a^{2}c^{2}+1}+\sqrt{c^{4}+2b^{2}a^{2}+1}\geq 6$.

Solutie:

$$\left(x-y\right)^{2}\geq 0⇒x^{2}+y^{2}\geq 2xy$$

$$\left(a^{4}+1\right)+2b^{2}c^{2}\geq 2a^{2}+2b^{2}c^{2}=a^{2}+b^{2}c^{2}+\left(a^{2}+\left(bc\right)^{2}\right)\geq a^{2}+b^{2}c^{2}+2abc==\left(a+bc\right)^{2}⇒$$

$$⇒\sqrt{a^{4}+2b^{2}c^{2}+1}\geq a+bc⇒$$

$$⇒\sqrt{a^{4}+2b^{2}c^{2}+1}+\sqrt{b^{4}+2a^{2}c^{2}+1}+\sqrt{c^{4}+2b^{2}a^{2}+1}\geq ab+bc+ca+a+b+c$$

$$\left(a+b+c\right)^{2}=a^{2}+b^{2}+c^{2}+2\left(ab+bc+ca\right)\geq 3\left(ab+bc+ca\right)=9⇒$$

$$⇒a+b+c\geq 3$$

3. Demonstrati ca pentru orice numere rationale $x,y,z$ diferite doua cate doua, numarul

$A=\sqrt{\frac{1}{\left(x-y\right)^{2}}+\frac{1}{\left(y-z\right)^{2}}+\frac{1}{\left(z-x\right)^{2}}}\in Q$.

Solutie:

$$a=x-y , b=y-z⇒x-z=a+b$$

$$A^{2}=\frac{1}{a^{2}}+\frac{1}{b^{2}}+\frac{1}{\left(a+b\right)^{2}}=\frac{\left(a+b\right)^{2}\left(a^{2}+b^{2}\right)+a^{2}b^{2}}{a^{2}b^{2}\left(a+b\right)^{2}}=\frac{\left(a^{2}+b^{2}\right)^{2}+2ab\left(a+b\right)+a^{2}b^{2}}{a^{2}b^{2}\left(a+b\right)^{2}}=$$

$=\frac{\left(a^{2}+b^{2}+ab\right)^{2}}{a^{2}b^{2}\left(a+b\right)^{2}}$ .

4. Fie $a,b\in Z si \left|a\right|\geq \left|b\right|$. Aratati ca

$\sqrt{a^{2}+\left(b-1\right)^{2}+\left(a^{2}-b\right)^{2}}\in N$ daca si numai daca $\left|a\right|=\left|b\right|$.

Solutie:

Daca $\sqrt{a^{2}+\left(b-1\right)^{2}+\left(a^{2}-b\right)^{2}}\in N⇒a^{2}+\left(b-1\right)^{2}+\left(a^{2}-b\right)^{2}=n^{2}$

$$\left|a\right|\geq \left|b\right|⇒a^{2}>0 sau \left(b-1\right)^{2}>0⇒n^{2}>\left(a^{2}-b\right)^{2}⇒$$

$$⇒n>a^{2}-b$$

$$n^{2}=a^{2}+b^{2}-2b+1+a^{4}-2a^{2}b+b^{2}=\left(a^{2}-b+1\right)^{2}+b^{2}-a^{2}\leq \left(a^{2}-b+1\right)^{2}⇒$$

$$⇒n\leq a^{2}-b+1$$

Deoarece $n\in N$ rezulta ca
$n=a^{2}-b+1$, deci $b^{2}-a^{2}=0$.

Daca $\left|a\right|=\left|b\right|$ atunci $\sqrt{a^{2}+\left(b-1\right)^{2}+\left(a^{2}-b\right)^{2}}=\sqrt{\left(a^{2}-b+1\right)^{2}}\in N$.

5. Daca $x,y,z,a$ sunt numere reale pozitive astfel incat $\sqrt{x}+\sqrt{y}+\sqrt{z}=a si \frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}=\frac{1}{a}$ atunci sa se arate ca $x=a^{2} sau y=a^{2} sau z=a^{2}$.

Solutie:

$$\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}=\frac{1}{\sqrt{x}+\sqrt{y}+\sqrt{z}}⇒\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)\left(\sqrt{x}+\sqrt{y}+\sqrt{z}\right)=\sqrt{xyz}⇒$$

$$⇒\sqrt{xy}\left(\sqrt{x}+\sqrt{y}\right)+\sqrt{z}\left(\sqrt{x}+\sqrt{y}\right)^{2}+z\left(\sqrt{x}+\sqrt{y}\right)=0⇒$$

$$⇒\left(\sqrt{x}+\sqrt{y}\right)\left(\sqrt{y}+\sqrt{z}\right)\left(\sqrt{z}+\sqrt{x}\right)=0⇒$$

$⇒\sqrt{x}+\sqrt{y}=0 sau \sqrt{y}+\sqrt{z}=0 sau \sqrt{z}+\sqrt{x}=0⇒\sqrt{z}=a sau \sqrt{x}=a sau \sqrt{y}=a$.

6. Sa se determine numerele reale $x$ pentru care are loc egalitatea:

$$\frac{1}{1006}\left(\sqrt{2\sqrt{2}x-x^{2}+1^{2}-2}+\sqrt{2\sqrt{2}x-x^{2}+2^{2}-2}+…+\sqrt{2\sqrt{2}x-x^{2}+2011^{2}-2}\right)==2011$$

Solutie:

$$\sqrt{1^{2}-\left(x-\sqrt{2}\right)^{2}}+\sqrt{2^{2}-\left(x-\sqrt{2}\right)^{2}}+…+\sqrt{2011^{2}-\left(x-\sqrt{2}\right)^{2}}=1006∙2011$$

$$\sqrt{k^{2}-\left(x-\sqrt{2}\right)^{2}}\leq k , k\in \left\{1,2,3,…,2011\right\}⇒$$

$$⇒\sqrt{1^{2}-\left(x-\sqrt{2}\right)^{2}}+\sqrt{2^{2}-\left(x-\sqrt{2}\right)^{2}}+…+\sqrt{2011^{2}-\left(x-\sqrt{2}\right)^{2}}\leq \leq 1+2+3+…+2011=1006∙2011⇒$$

$$⇒\sqrt{k^{2}-\left(x-\sqrt{2}\right)^{2}}=k , k\in \left\{1,2,3,…,2011\right\}⇒x=\sqrt{2}$$

7. Determinati numerele rationale $a si b$ astfel incat $\sqrt{2}+b\sqrt{3}=2\sqrt{a}+3\sqrt{b}$ .

Solutie:

Prin ridicarea relatiei la patrat se obtine:

$$2a^{2}+3b^{2}-4a-9b+2ab\sqrt{6}=12\sqrt{ab}$$

$$2a^{2}+3b^{2}-4a-9b=x$$

Prin ridicarea relatiei la patrat se obtine:

$$\left(x+2ab\sqrt{6}\right)^{2}=144ab⇒$$

$$⇒4xab\sqrt{6}+\left(x^{2}+24a^{2}b^{2}-144ab\right)=0⇒$$

$$⇒4ab\left(2a^{2}+3b^{2}-4a-9b\right)=0 , 144ab-2a^{2}-3b^{2}+4a+9b-24a^{2}b^{2}=0⇒$$

$⇒\left(a,b\right)\in \left\{\left(0,0\right),\left(0,3\right),\left(2,0\right),(2,3)\right\}$.

8. Sa se arate ca oricare ar fi trei numere naturale impare exista un numar natural impar astfel incat suma patratelor celor patru numere sa fie patrat perfect.

Solutie:

$$x=2a+1,y=2b+1,z=2c+1$$

$$x^{2}+y^{2}+z^{2}=4\left(a^{2}+b^{2}+c^{2}+a+b+c\right)+2+1=2m+1=\left(m+1\right)^{2}-m^{2}$$

$$m=2\left(a^{2}+b^{2}+c^{2}+a+b+c\right)+1$$

9. Sa se determine toate perechile de numere reale $\left(x,y\right)$ care satisfac simultan conditiile:

a)$x\geq y\geq 1$;

b)$2x^{2}-xy-5x+y+4=0$.

Solutie:

$$a=x-1\geq 0 , b=y-1\geq 0 , a\geq b$$

$$2\left(a+1\right)^{2}-\left(a+1\right)\left(b+1\right)-5\left(a+1\right)+(b+1)+4=0⇒$$

$$⇒2a^{2}-2a-ab+1=0⇒\left(a-1\right)^{2}+a\left(a-b\right)=0⇒$$

$⇒a-1=0 , a-b=0⇒\left(x,y\right)=\left(2,2\right)$.

10. Numerele $n\in N^{\*} si x\_{1},x\_{2},…,x\_{n}\in Z$ satisfac relatia:

$x\_{1}^{2}+x\_{2}^{2}+…+x\_{n}^{2}+n^{3}\leq \left(2n-1\right)\left(x\_{1}+x\_{2}+…+x\_{n}\right)+n^{2}$.

Sa se arate ca:

a)$x\_{1},x\_{2},…,x\_{n}\in N$;

b)numarul $x\_{1}+x\_{2}+…+x\_{n}+n+1$ nu este patrat perfect.

Solutie:

$$\sum\_{k=1}^{n}x\_{k}^{2}-2n\sum\_{k=1}^{n}x\_{k}+n∙n^{2}+\sum\_{k=1}^{n}x\_{k}-n∙n\leq 0⇒$$

$$⇒\sum\_{k=1}^{n}\left(x\_{k}-n\right)^{2}+\sum\_{k=1}^{n}\left(x\_{k}-n\right)\leq 0⇒\sum\_{k=1}^{n}\left(x\_{k}-n\right)(x\_{k}-n+1)\leq 0$$

Deoarece produsul a doua numere intregi consecutive este un numar pozitiv rezulta ca

$$\left(x\_{k}-n\right)\left(x\_{k}-n+1\right)=0⇒x\_{k}\in \left\{n-1,n\right\}$$

$$n(n-1)\leq \sum\_{k=1}^{n}x\_{k}\leq n^{2}⇒$$

$$⇒n^{2}<n^{2}+1\leq 1+n+\sum\_{k=1}^{n}x\_{k}\leq n^{2}+n+1<\left(n+1\right)^{2}$$

Rezulta ca $x\_{1}+x\_{2}+…+x\_{n}+n+1$ nu este patrat perfect.

11. Fie $x,y,z,t$ numere natural distinct si prime intre ele doua cate doua. Sa se demonstreze ca daca

$\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{t}$ ,

atunci $x,y,z,t$ sunt patrate perfecte.

Solutie:

$$\sqrt{x}+\sqrt{y}=\sqrt{t}-\sqrt{z}⇒x+y+2\sqrt{xy}=t+z-2\sqrt{tz}⇒$$

$$⇒2\left(\sqrt{xy}+\sqrt{tz}\right)=t+z-x-y⇒\sqrt{xyzt}=\frac{1}{8}\left[\left(z+t-x-y\right)^{2}-4\left(xy+zt\right)\right]\in Q$$

Rezulta ca $\sqrt{xyzt}\in N⇒x,y,z,t se descompun in produse de factori primi cu exponenti numere pare ⇒x,y,z,t sunt patrate perfecte$.

12. Demonstrati ca numarul $A=\sqrt{3\left[\left(n^{2}+n-3\right)\left(n^{2}+n-7\right)+5\right]}$ este irational oricare ar fi $n\in Z$.

Solutie:

$$\left(n^{2}+n-3\right)\left(n^{2}+n-7\right)+5=\left(n^{2}+n\right)^{2}-10\left(n^{2}+n\right)+25+1=\left(n^{2}+n-5\right)^{2}+1$$

 $A=\sqrt{3\left[\left(n^{2}+n-5\right)^{2}+1\right]}=\sqrt{3\left(k^{2}+1\right)}$

$$k=3p⇒k^{2}+1=9p^{2}+1⇒3∤\left(k^{2}+1\right)$$

$$k=3p+1⇒k^{2}+1=3\left(3p^{2}+2p\right)+2⇒3∤\left(k^{2}+1\right)$$

$$k=3p+2⇒k^{2}+1=3\left(3p^{2}+4p+1\right)+2⇒3∤\left(k^{2}+1\right)$$

13. Aratati ca daca $a,n\in N$ atunci $\sqrt{a\left(a+n\right)\left(a+2n\right)\left(a+3n\right)+n^{4}}\in N$.

Solutie:

$$a\left(a+n\right)\left(a+2n\right)\left(a+3n\right)+n^{4}=x\left(x+2n^{2}\right)+n^{4}=\left(x+n^{2}\right)^{2} , x=a^{2}+3an$$

Exercitii propuse ca tema pentru elevi:

1. Demonstrati ca daca $n\in N , n\geq 3$ atunci

$S=\sqrt{1+\frac{1}{2^{2}}+\frac{1}{3^{2}}}+\sqrt{1+\frac{1}{3^{2}}+\frac{1}{4^{2}}}+…+\sqrt{1+\frac{1}{\left(n-1\right)^{2}}+\frac{1}{n^{2}}}\in Q$.

Indicatie:

$$1+\frac{1}{\left(k-1\right)^{2}}+\frac{1}{k^{2}}=\frac{k^{4}-2k^{3}+3k^{2}-2k+1}{\left(k-1\right)^{2}k^{2}}=\frac{\left(k^{2}-k+1\right)^{2}}{\left(k-1\right)^{2}k^{2}}$$

$$\frac{k^{2}-k+1}{(k-1)k}=1+\frac{1}{k-1}-\frac{1}{k}$$

2. Determinati $n\in N$ astfel incat $\sqrt{n^{2}-n+19}\in N$.

Indicatie:

$$n^{2}-n+19=k^{2}⇒4n^{2}-4n+76=4k^{2}⇒4k^{2}-\left(2n-1\right)^{2}=75$$

3. Determinati partea intreaga a numarului

$A=1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+…+\frac{1}{\sqrt{2601}}$ .

Indicatie:

$$2\left(\sqrt{n+1}-\sqrt{n}\right)=\frac{2}{\sqrt{n+1}+\sqrt{n}}<\frac{2}{\sqrt{n}+\sqrt{n}}=\frac{1}{\sqrt{n}}$$

$$2\left(\sqrt{n}-\sqrt{n-1}\right)=\frac{2}{\sqrt{n}+\sqrt{n-1}}>\frac{1}{\sqrt{n}}$$

4. Daca $a,b,c$ sunt numere intregi nenule, $\ne c si \frac{a}{c}=\frac{a^{2}+b^{2}}{b^{2}+c^{2}}$ , atunci $a^{2}+b^{2}+c^{2}$ este numar natural compus.

Indicatie:

$$\frac{a}{c}=\frac{a^{2}+b^{2}}{b^{2}+c^{2}}⇒\left(a-c\right)\left(b^{2}-ac\right)=0⇒b^{2}=ac$$

$$a^{2}+b^{2}+c^{2}=a^{2}+2ac+c^{2}-b^{2}=\left(a+c-b\right)(a+c+b)$$

$$a^{2}+b^{2}+c^{2}\in N^{\*} , a^{2}+b^{2}+c^{2}>3$$

Daca $a^{2}+b^{2}+c^{2}$ ar fi numar prim atunci:

1)$a+c-b=1$ si $a+c+b=a^{2}+b^{2}+c^{2}$:

2)$ a+c+b$ si $a+c-b=a^{2}+b^{2}+c^{2}$;

3)$ a+c-b=-1$ si $a+c+b=-\left(a^{2}+b^{2}+c^{2}\right)$;

4)$ a+c+b=-1$ si $a+c-b=-\left(a^{2}+b^{2}+c^{2}\right)$.

Rezulta ca

$$\pm 2\left(a+c\right)+a^{2}+b^{2}+c^{2}+1=0⇒\left(a\pm 1\right)^{2}+\left(c\pm 1\right)^{2}+b^{2}=1⇒$$

$⇒a=c=-1$ sau $a=c=1$ imposibil.